Hamiltonians for the Quantum Hall Effect on Spaces with Non-Constant Metrics

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The problem of studying the quantum Hall effect on manifolds with non constant metric is addressed. The Hamiltonian on a space with hyperbolic metric is determined, and the spectrum and eigenfunctions are calculated in closed form. The hyperbolic disk is also considered and some other applications of this approach are discussed as well.

1. INTRODUCTION

The problem of investigating the quantum Hall effect (QHE) on different types of manifold has been of great interest recently. Some of the previous work in this area originated in the course of generalizing the Hall current from the $SO(3)$ two-sphere S^2 to the four-sphere S^4 of the invariance group $SO(5)$ (Zhang and Hu, 2001). Quantum Hall droplets have been considered as well on complex projective spaces CP^d (Karabali and Nair, 2002, 2004). The idea all along has been to generalize the Landau problem on different types of higher dimensional spaces. This kind of work combines the study of several important areas in mathematics and physics.

Zhang and Hu (2001) considered the Landau problem for charged fermions on S^4 under the influence of a background magnetic field which is related to the standard $SU(2)$ instanton. In ordinary quantum Hall effects, a droplet of fermions occupying a certain region behaves as an incompressible fluid, a characteristic property of the QHE, the low energy excitations being area-preserving deformations which behave as massless chiral bosons. Some work on chiral boson theories related to the QHE has been described in Bracken (2001). Now S^4 may be considered since the edge excitations could lead to higher spin massless fields, in

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particular, the graviton. The original work of Hu and Zhang (2002) has been continued to include the QHE on even-dimensional complex projective spaces CP^k, for example. This is interesting since one can obtain incompressible droplet states by coupling the fermions to a $U(1)$ background field.

The Landau problem has been of fundamental importance to the QHE from the beginning (Ezawa, 2000; Khare, 2005). The main aim here is to consider generalizations of the QHE to spaces with nonflat metrics by considering Hamiltonians of the Laplace-Beltrami form. For example, the spectrum of the Laplacian on the ball B^d in different dimensions has been investigated. Of interest here will be the generalization of the Landau problem on the plane to the case of spaces with non-constant metrics. The Hall conductance can be thought of as a type of curvature, and it would be useful to understand this relationship more thoroughly and in other frameworks. We begin by introducing a classical model of a charged particle in a magnetic field on the hyperbolic plane (Comtet, 1987). The metric in this case is diagonal but non-constant. Next the quantum version of the Landau problem on the plane will be reviewed for the sake of completeness, and most of the important results will be derived (Chandelier *et al.*, 2004; Iengo and Li, 1994; Jellal, 2005). The quantized version for the related problem under a diagonal but non-constant hyperbolic metric will be formulated next. The Hamiltonian can be derived by making use of the Laplace-Beltrami operator for a particle of mass m on a manifold with metric g_{ij} under a monopole field. A separation of variables solution can be obtained and the associated eigenvalues and eigenfunctions for this differential equation can be written down explicitly. The same problem can then be formulated from the Lie algebraic point of view as well. The Hamiltonian can be expressed in terms of the relevant Casimir operator, and the energies can be obtained in this way as well. Some interesting observations with regard to the operator ordering question for the Hamiltonian as well as the underlying symmetry vector fields can be made. Finally, this will be repeated for the case of formulating a Hamiltonian for the hyperbolic disk.

2. CLASSICAL LAGRANGIAN FORMULATION IN A HYPERBOLIC SPACE

Before investigating the quantum dynamics of a charged particle under the influence of a magnetic field in a space with a non-constant metric, it is worth reviewing the classical formulation of the dynamics of a charged particle in a space with a hyperbolic metric. Consider dynamics then on the upper half of the complex plane, the Poincaré plane defined by,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C}, \, y > 0\}. \tag{2.1}$$

The metric on this space can be written as

$$ds^2 = \frac{a^2}{y^2}(dx \otimes dx + dy \otimes dy).$$ (2.2)

The scalar curvature for this space is a negative constant $K = -2/a^2$. A constant magnetic field can be obtained from a vector potential in the plane given by

$$\mathbf{A} = \left(-\frac{\beta}{y}, 0\right), \qquad \beta = Ba^2.$$ (2.3)

The quantity β defined in (2.3) is frequently referred to as the rescaled field.

The Euler-Lagrange equations which determine the classical trajectories of a charged particle in the field generated by $\mathbf{A}$ can be determined from the Lagrangian

$$L = \frac{a^2}{y^2}(\dot{x}^2 + \dot{y}^2) - \beta\frac{\dot{x}}{y}.$$ (2.4)

The differentiation in (2.4) is with respect to the time variable. The Hamiltonian is determined from L by means of

$$H = \sum_{r=1}^{N} p_r \dot{q}_r - L,$$ (2.5)

where

$$p_x = p_1 = \frac{\partial L}{\partial \dot{x}} = \frac{2a^2}{y^2}\dot{x} - \frac{B}{y}a^2, \qquad p_y = p_2 = \frac{\partial L}{\partial \dot{y}} = \frac{2a^2}{y^2}\dot{y}.$$ (2.6)

Solving this pair individually for $\dot{x}$ and $\dot{y}$, the results are substituted into (2.5) to obtain

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} - \frac{a^2}{y^2}(\dot{x}^2 + \dot{y}^2) + \beta\frac{\dot{x}}{y} \\
&= p_x\left(\frac{y^2}{2a^2}p_x + \frac{y}{2}B\right) + p_y\frac{y^2}{2a^2}p_y - \frac{a^2}{y^2}\left(\frac{y^2}{2a^2}p_x + \frac{y}{2}B\right)^2 \\
&\quad - \frac{y^2}{4a^2}p_y^2 + \frac{\beta}{y}\left(\frac{y^2}{2a^2}p_x + \frac{y}{2}B\right) \\
&= \frac{1}{4a^2}\left\{y^2(p_x^2 + p_y^2) + 2\beta y p_x + \beta^2\right\}.
\end{aligned}$$ (2.7)

Constants of the motion are determined by Noether's theorem and are given by

$$\begin{aligned}
L_1 &= xp_x + yp_y, \\
L_2 &= p_y, \\
L_3 &= (y^2 - x^2)p_x - 2xyp_y + 2\beta y.
\end{aligned}$$ (2.8)

Classically the relative order of each of the factors in the L_j in (2.8) is irrelevant. Thus, by direct calculation, it follows that

$$L_2 L_3 + L_1^2 = y^2 \left(p_x^2 + p_y^2 \right) + 2\beta y p_x.$$

By comparison with (2.7), it follows that

$$H = \frac{1}{4a^2} \left(L_2 L_3 + L_1^2 + \beta^2 \right). \tag{2.9}$$

By resorting to Hamiltonian (2.7), Hamilton's equations can be formulated directly as

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{2a^2} (y^2 p_x + \beta y), \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{y^2}{2a^2} p_y,$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = \frac{1}{2a^2} \left(y \left(p_x^2 + p_y^2 \right) + \beta p_x \right). \tag{2.10}$$

The Poisson bracket of any two classical functions F and G is given by

$$\{F, G\} = \sum_{r=1}^{N} \left(\frac{\partial F}{\partial q^r} \frac{\partial G}{\partial p_r} - \frac{\partial F}{\partial p_r} \frac{\partial G}{\partial q^r} \right). \tag{2.11}$$

By straightforward differentiation of the L_j in (2.8), it can be verified that, with the canonical Poisson bracket, the L_j given in (2.8) generate an $sl(2, \mathbb{R})$ Lie algebra

$$\{L_1, L_2\} = L_2, \qquad \{L_1, L_3\} = -L_2, \qquad \{L_2, L_3\} = 2L_1. \tag{2.12}$$

Eliminating p_x and p_y from (2.8), the classical path can be determined, and with respect to the Euclidean plane, it describes a circle.

3. QUANTUM PROBLEM IN THE PLANE

The dynamics of a charged particle in the plane under the influence of an external uniform magnetic field B which is oriented at right angles to the plane will be studied first. This will provide a setting in which to introduce the QHE as well. It will be useful here to frequently use complex coordinates z and $\bar{z}$ defined by $z = x + iy$ and $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. The gauged Hamiltonian is usually written in the form

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2. \tag{3.1}$$

In the symmetric gauge and formulated in terms of complex z and $\bar{z}$, (3.1) can be written

$$H = -\frac{2\hbar^2}{m} \bar{\partial} \partial + \frac{m\omega_0^2}{8} |z|^2 - \frac{\hbar \omega_c}{2} (z\partial - \bar{z}\bar{\partial}), \tag{3.2}$$

where $\omega_c = e|B|/mc$ is the cyclotron frequency.

Introduce two quantum operators a and $a^\dagger$ defined by,

$$a = -2i\sqrt{\frac{\hbar}{2m\omega_c}}\left(\bar{\partial} + \frac{m\omega_c}{4\hbar}z\right), \qquad a^\dagger = -2i\sqrt{\frac{\hbar}{2m\omega_c}}\left(\partial - \frac{m\omega_c}{4\hbar}\bar{z}\right). \quad (3.3)$$

Using these, we calculate

$$aa^\dagger - a^\dagger a = 1, \tag{3.4}$$

and

$$\frac{\hbar\omega_c}{2}(aa^\dagger + a^\dagger a) = -\frac{2\hbar^2}{m}\partial\bar{\partial} - \frac{\hbar\omega_c}{2}(z\partial - \bar{z}\bar{\partial}) + \frac{1}{8}m\omega_c^2|z|^2. \tag{3.5}$$

Therefore, in terms of the operators defined by (3.3), we can write

$$H = \frac{1}{2}\hbar\,\omega_c(2a^\dagger a + 1). \tag{3.6}$$

Now $a^\dagger a$ can be interpreted as a number operator, thus defining a number basis $|n\rangle$, the energy spectrum of H is given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \qquad n = 0, 1, 2, \ldots. \tag{3.7}$$

The Landau levels E_n are degenerate with respect to the center of Larmor's circular orbits. Corresponding eigenfunctions are obtained from the ground state eigenfunction as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{3.8}$$

where the ground state $|0\rangle$ obeys the equation

$$a|0\rangle = 0. \tag{3.9}$$

Substituting a from (3.3), this implies a first order equation for the ground state function given by

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{m\omega_c}{4\hbar}z\right)\psi_0(z, \bar{z}) = 0. \tag{3.10}$$

This equation has the general solution

$$\psi_0(z, \bar{z}) = f(z)\exp\left(-\frac{|z|^2}{4z_0^2}\right), \tag{3.11}$$

where $z_0 = \sqrt{\hbar c/eB}$.

The Hamiltonian in (3.2) can be generalized to a many-particle system described by the total Hamiltonian

$$H = \sum_{i=1}^{N} \left\{ -\frac{2\hbar^2}{m} \bar{\partial}_i \partial_i - \frac{\hbar\omega_c}{2}(z_i \partial_i - \bar{z}_i \bar{\partial}_i) + \frac{m\omega_c^2}{8}|z_i|^2 \right\}. \qquad (3.12)$$

Suppose we consider N-particles in the lowest Landau level, which means that the quantum numbers satisfy $n_i = 0$ with $i = 1, 2 \ldots, N$ and each n_i corresponds to the spectrum of (3.6). The total wavefunction can be written in terms of the Slater determinant

$$\Psi(z, \bar{z}) = \epsilon^{i_1 \ldots i_N} z_{i_1}^{n_1} \ldots z_{i_N}^{n_N} \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{4z_0^2} \right), \qquad (3.13)$$

where $\epsilon^{i_1 \ldots i_N}$ is the fully antisymmetric tensor and the n_i are integers. Expanded as a Vandermonde determinant, we have

$$\Psi_1(z, \bar{z}) = K \prod_{i,j}(z_i - z_j) \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{4z_0^2} \right). \qquad (3.14)$$

This can be compared with the Laughlin wavefunction given by

$$\Psi_m(z, \bar{z}) = K_m \prod_{i,j}(z_i - z_j)^m \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{4z_0^2} \right). \qquad (3.15)$$

This matches (3.14) when $m = 1$ and is a good ansatz to describe the fractional QHE at the filling factor $\nu = 1/m$.

Let us mention that the filling factor in units such that he/c is one can be written as

$$\nu = \frac{2\pi \mathcal{N}}{B}, \qquad (3.16)$$

where $\mathcal{N}$ is the density of particles

$$\mathcal{N} = \frac{N}{S},$$

and S is the plane surface area. In the QHE, ν in (3.16) must be quantized and reads

$$\nu = \frac{N}{N_\phi}. \qquad (3.17)$$

This can be either an integer or fractional depending on which kind of effect is involved and N_ϕ is the quantum flux number. Thus, the magnetic flux is quantized here.

4. THE QUANTUM PROBLEM IN A HYPERBOLIC GEOMETRY

A single particle quantum Hamiltonian can be developed for the Poincaré half-plane. It can be derived from the electromagnetically gauged form of the Laplace-Beltrami operator, such that the contribution of the vector potential has been included. In this section, units are taken such that $\hbar = c = e = 1$. If the particle has a mass m on a space with metric g_{ij}, the operator can be written as

$$H^{LB} = \frac{1}{2m\sqrt{g}}(\mathbf{p} - \mathbf{A})_i (\sqrt{g} g^{ij})(\mathbf{p} - \mathbf{A})_j, \tag{4.1}$$

where g in (4.1) is the determinant of the metric, g^{ij} the inverse of g_{ij}, and we follow de Witt's prescription (De Witt, 1957) such that the covariant derivative $\mathbf{p}$ contains a contribution which is directly related to the metric. The metric for the space which is of interest here is given by (2.2), and the gauge is fixed by taking the vector potential in the plane to have the form (2.3). From the form of the metric, it is clear that $\sqrt{g} = a^2/y^2$ and the inverse metric has elements which are the reciprocals of those in g_{ij}. When the metric has diagonal form, the Laplace-Beltrami operator takes the classical form

$$H^{LB} = \frac{y^2}{2ma^2}\left(P_1^2 + P_2^2\right), \tag{4.2}$$

where the momenta P_j are gauged with the electromagnetic contributions due to a nontrivial vector potential, which can be written

$$P_j = p_j - A_j. \tag{4.3}$$

When H^{LB} is quantized, the form for H^{LB} given in (4.2) could be adopted, or the operators y could be placed in a different order

$$H^{LB} = \frac{1}{2ma^2}\, y\left(P_1^2 + P_2^2\right)y, \tag{4.4}$$

or even with the factor of y^2 placed entirely to the far right of the operator. These different cases just correspond to the usual operator ordering ambiguities which arise during quantization. Some remarks related to this will be made later.

To finish the canonical quantization procedure following de Witt's prescription, the momentum operators p_j have to be constructed, and these contain a contribution which is related to the metric determinant g, and are given as follows

$$p_1 = p_x = -i\left(\partial_x + \tfrac{1}{2}\partial_x \ln \sqrt{g}\right) = -i\partial_x,$$

$$p_2 = p_y = -i\left(\partial_y + \tfrac{1}{2}\partial_y \ln \sqrt{g}\right) = -i\partial_y + \frac{i}{y}. \tag{4.5}$$

Substituting p_j given in (4.5) into the Hamiltonian (4.4), we obtain that

$$H^{LB} = \frac{1}{2ma^2} y \left\{ \left(-i\partial_x + \frac{\beta}{y} \right)^2 + \left(-i\partial_y + \frac{i}{y} \right)^2 \right\} y, \qquad (4.6)$$

and β is defined in (2.3). Expanding the terms in this operator out in full, we find

$$y \left(-i\partial_x + \frac{\beta}{y} \right)^2 y = -y^2 \partial_x^2 - 2i\beta y \partial_x + \beta^2,$$

$$(4.7)$$

$$y \left(-i\partial_y + \frac{i}{y} \right) (-i\partial_y y + i) = -iy \left(-i\partial_y + \frac{i}{y} \right) y \partial_y = -y^2 \partial_y^2.$$

Therefore, the Hamiltonian (4.4) takes the form,

$$H^{LB} = \frac{1}{2ma^2} \left\{ -y^2 (\partial_x^2 + \partial_y^2) - 2i\beta y \partial_x + \beta^2 \right\}, \qquad (4.8)$$

and the associated eigenvalue problem takes the form

$$H^{LB} \Psi = E \Psi. \qquad (4.9)$$

Just as in the case of (4.4), the three constants of the motion L_j in (1.6) are also determined up to ordering ambiguities as well. The exact form of the operators can be established in this case by using them to establish a particular Lie algebra structure such that the associated Casimir operator matches the Hamiltonian (4.8) up to constant terms. In this case remarkably, combining these approaches seems to be sufficient to eliminate the ordering ambiguities, that is, the ordering given in the quantum form of the L_j is the one which matches (4.4), and the other cases for the L_j and H^{LB} with y in different orientations will not coincide under this procedure. Consider the following form for the operators L_j,

$$L_1 = -i\partial_x x + y \left(-i\partial_y + \frac{i}{y} \right),$$

$$L_2 = -i\partial_x,$$

$$L_3 = -i \partial_x (y^2 - x^2) - 2xy \left(-i\partial_y + \frac{i}{y} \right) + 2\beta y. \qquad (4.10)$$

Moving all the operator terms to the right-hand side of the variables by expanding out, these become

$$L_1 = -i(x \partial_x + y \partial_y),$$

$$L_2 = -i\partial_x,$$

$$L_3 = -i(y^2 - x^2) \partial_x + 2ixy\partial_y + 2\beta y. \qquad (4.11)$$

The calculations which can easily be carried out by using MAPLE (Char *et al.*, 1991) have been done this way. It is found that the operators in (4.11) satisfy the following commutation relations

$$[L_1, L_2] = i L_2, \qquad [L_1, L_3] = -i L_3, \qquad [L_2, L_3] = 2i L_1. \qquad (4.12)$$

The following combinations of the operators L_j can be written down

$$J_0 = \frac{1}{2}(L_2 - L_3), \qquad J_1 = \frac{1}{2}(L_2 + L_3), \qquad J_2 = L_1. \qquad (4.13)$$

If the L_j satisfy (4.12), it is easy to show using the properties of the commutator that the J_i defined in (4.13) satisfy the canonical form of the $su(1, 1)$ algebra described by the commutators (Barut and Raczka, 1986; Chen *et al.*, 2002)

$$[J_0, J_1] = i J_2, \qquad [J_0, J_2] = -i J_1, \qquad [J_1, J_2] = -i J_0. \qquad (4.14)$$

The Casimir operator which corresponds to these operators under this algebra is given by

$$C = J_0^2 - J_1^2 - J_2^2. \qquad (4.15)$$

Using (4.13), this can be written in terms of the L_j as follows

$$C = -L_2 L_3 - L_1^2 + i L_1. \qquad (4.16)$$

Again, by direct calculation, it can be verified using (4.11) that

$$-C = L_2 L_3 + L_1^2 - i L_1 = -y^2(\partial_x^2 + \partial_y^2) - 2i\beta y \partial_x.$$

Hence in terms of the Casimir operator C, the Hamiltonian (4.8) can be written in the equivalent form

$$H^{LB} = \frac{1}{2ma^2}(-C + \beta^2). \qquad (4.17)$$

The spectrum of H^{LB} can be obtained from the representation theory that corresponds to the operator C. Consider a unitary, irreducible representation of the group as eigenstates of C as well as the compact generator J_0. Let us choose a basis $|j, m\rangle$ such that

$$C|j, m\rangle = j(j + 1)|j, m\rangle, \qquad (4.18)$$

and where m is an eigenvalue of J_0,

$$J_0|j, m\rangle = m|j, m\rangle. \qquad (4.19)$$

Using (4.17), the effect of the Hamiltonian on $|j, m\rangle$ can be determined

$$H^{LB}|j, m\rangle = \frac{1}{2ma^2}\left(\beta^2 - j(j + 1)\right)|j, m\rangle. \qquad (4.20)$$

Substituting into eigenvalue problem (4.9), and taking $j = l - b$ where l is an integer, the energy eigenvalues can be expressed in the form,

$$E_{\beta,l} = \frac{1}{2ma^2}\left(\beta^2 + \frac{1}{4} - \left(l - \beta + \frac{1}{2}\right)^2\right).\tag{4.21}$$

It will be useful to compare the result in (4.21) with that obtained by solving the eigenvalue problem (4.9) in differential form. Consider now (4.10) with H^{LB} given by (4.8). Let us consider obtaining a class of solutions which have a separation of variables form

$$\Psi(x, y) = \alpha(x)\varphi(y).\tag{4.22}$$

Substituting (4.22) into (4.9), the differential equation takes the form

$$-y^2\left(\frac{\alpha_{xx}}{\alpha} + \frac{\varphi_{yy}}{\varphi}\right) - 2i\beta y\frac{\alpha_x}{\alpha} + \beta^2 - 2ma^2 E = 0.\tag{4.23}$$

To decouple the x and y variables in (4.23), let us require that $\alpha_x = -ic\alpha$, where c is a real, positive constant hence we take

$$\alpha(x) = e^{-icx}.\tag{4.24}$$

The appearance of the imaginary unit in the exponential reduces the equation entirely to real form. Substituting (4.24) into (4.23), we obtain an equation in terms of φ

$$\frac{\varphi_{yy}}{\varphi} - c^2 + \frac{2\beta c}{y} + \frac{1}{y^2}(-\beta^2 + 2mE) = 0.\tag{4.25}$$

Introducing the variable to $s = 2cy$ and setting $-\beta^2 + 2mE = \frac{1}{4} - n^2$, Eq. (4.25) takes the form of a Whittaker equation

$$\frac{\varphi''}{\varphi} - \frac{1}{4} + \frac{\beta}{s} + \frac{1}{s^2}\left(\frac{1}{4} - n^2\right) = 0,\tag{4.26}$$

such that the energies are related to n through

$$E_n = \frac{1}{2ma^2}\left(\frac{1}{4} - n^2 + \beta^2\right).\tag{4.27}$$

The general form for the solutions to (4.26) can be written in terms of the confluent hypergeometric function

$$M_{\beta,n}(s) = e^{-s/2}s^{1/2+n}\,_1F_1\left(\frac{1}{2} + n - \beta; 1 + 2n; s\right),$$

$$M_{\beta,-n}(s) = e^{-s/2}s^{1/2-n}\,_1F_1\left(\frac{1}{2} - n - \beta; 1 - 2n; s\right).\tag{4.28}$$

If n is taken such that $n = \beta - l - \frac{1}{2}$, where l is chosen to be an integer such that $0 \leq l < \beta - \frac{1}{2}$, the confluent hypergeometric function in $M_{\beta,n}$ will truncate to the form of a Laguerre polynomial (Andrews *et al.*, 1999; Luke, 1969). Moreover, $M_{\beta,n}(2cy)$ is defined at $y = 0$ and square integrable in y on the domain (2.1). The connection between the hypergeometric function and Laguerre polynomial is provided by

$$L_n^{(\tau)}(z) = \frac{(\tau + 1)_n}{n!} {}_1F_1(-n; \tau + 1; z).$$

Substituting this into $M_{\beta,n}(s)$ in (4.28), the eigenfunctions of (4.9), up to a normalization constant can be written in the form

$$\Psi(x, y) = \mathcal{N} e^{-icx-cy} y^{\beta-l} L_l^{2\beta-2l-1}(2cy), \tag{4.29}$$

and the energies are given by (4.27) with $n = \beta - l - 1/2$ as,

$$E_{\beta,l} = \frac{1}{2ma^2}\left(\beta^2 + \frac{1}{4} - \left(l - \beta + \frac{1}{2}\right)^2\right). \tag{4.30}$$

These results for $E_{\beta,l}$ can be compared with the expression given in (4.21).

Note that (4.8) can be written in complex form as well

$$H^{LB} = \frac{1}{2ma^2}\{(z - \bar{z})^2 \bar{\partial}\partial - \beta(z - \bar{z})(\partial + \bar{\partial}) + \beta^2\}. \tag{4.31}$$

For equal mass particles, (4.31) can be generalized to the case of a many-particle system as was done in (3.12) to give the Hamiltonian

$$H^{LB} = \frac{1}{2ma^2} \sum_{i=1}^{N}[(z_i - \bar{z}_i)^2 \bar{\partial}_i \partial_i - \beta(z_i - \bar{z}_i)(\partial_i + \bar{\partial}_i) + \beta^2]. \tag{4.32}$$

5. RESULTS FOR OTHER GEOMETRIES AND CONCLUSIONS

A Hamiltonian will be developed using the method described in the last section for a system on the hyperbolic disk $\mathbb{B}_\rho^1$, which is defined to be

$$\mathbb{B}_\rho^1 = \{w = x + iy \in \mathbb{C} | |w|^2 < \rho^2\}, \tag{5.1}$$

which carries the Bergman-Kähler metric

$$ds^2 = \left(1 - \frac{|w|^2}{\rho^2}\right)^{-2}(dx \otimes dx + dy \otimes dy), \tag{5.2}$$

where $|w|^2 = x^2 + y^2$, so the metric is again diagonal. Define the function

$$\phi(x, y) = 1 - \frac{x^2 + y^2}{\rho^2}, \tag{5.3}$$

and clearly $\sqrt{g} = \phi^{-2}$ in this case. To write a Hamiltonian on $\mathbb{B}_\rho^1$, we work out (4.1) with the metric (5.2) and take a vector potential with two components which is defined everywhere on (5.1). For x and y such that $|w|^2 < \rho$, then with B the magnetic field in a symmetric gauge let

$$\mathbf{A} = B(y, -x). \tag{5.4}$$

Using the definitions in (4.5), the vector potential is specified by

$$p_x = p_1 = -i\left(\partial_x + \frac{2x}{\rho^2\phi}\right), \qquad p_y = p_2 = -i\left(\partial_y + \frac{2y}{\rho^2\phi}\right). \tag{5.5}$$

Therefore, substituting (5.4) and (5.5) into (4.1), we have

$$
\begin{aligned}
H &= \frac{1}{2m}\phi^2\left(\left(-i\partial_x - \frac{2ix}{\rho^2\phi} - By\right)^2 + \left(-i\partial_y - \frac{2iy}{\rho^2\phi} + Bx\right)^2\right) \\
&= \frac{\phi}{2m}\left\{-\phi(\partial_x^2 + \partial_y^2) - \frac{4}{\rho^2}(x\partial_x + y\partial_y) + 2iB\phi(y\partial_x - x\partial_y)\right. \\
&\quad \left. + B^2\phi - \frac{4}{\rho^2}\left(1 + \frac{2|w|^2}{\rho^2\phi}\right)\right\}.
\end{aligned}
\tag{5.6}
$$

As a last application of geometrical methods to generating Hamiltonians for this area, consider the approach proposed by Haldane (1983) to overcoming the symmetry problem in the Laughlin theory of the fractional QHE described by wavefunction (3.15) at the filling fraction $\nu = 1/m$. To phrase the problem more precisely, (3.15) is rotationally invariant due to angular momenta, but it is not translationally invariant. By considering particles living on a two-sphere in a magnetic monopole, Haldane formulated a theory that has all the symmetries and generalizes the Laughlin proposal.

The link with the two-sphere S^2 can be done by means of the following approach. First S^2 can be realized on the disk

$$\partial\mathbb{B}_\rho^1 = S^2 = \{w \in \mathbb{C} | |w| = \rho\},$$

which is the boundary of $\mathbb{B}_\rho^1$, and the basic features of $\mathbb{B}_\rho^1$ lead to those of S^2. The space H is invariant on the symmetric space $SU(1, 1)/U(1)$ and the projective space CP^1 can be obtained as

$$CP^1 = SU(2)/U(1).$$

The S^2 can be regarded as an analytic continuation of $SU(1, 1)$ to $SU(2)$. This suggests that the spectrum obtained on $\mathbb{B}_\rho^1$ is similar to the Landau problem on the sphere, except that the eigenfunctions should be invariant under the group $U(1)$.

The CP^1 expression shows that functions on S^2 can be thought of as functions of $SU(2)$, invariant under the $U(1)$ subgroup. A basis of functions for $SU(2)$ is given by the Wigner D-functions, and a basis for functions on S^2 is given by the $SU(2)$ Wigner functions $D^{(j)}_{L_3 R_3}(g)$ with trivial right $U(1)$ action, that is, the $U(1)_R$ charge $R_3 = 0$. Derivatives on S^2 can be identified as $SU(2)$ right rotations $SU(2)_R$, which satisfy an $SU(2)$ algebra. Consequently, covariant derivatives $D_\pm$ can be written as

$$D_+ = \frac{1}{\rho} L_2, \qquad D_- = \frac{1}{\rho} L_3. \qquad (5.7)$$

The operators in (5.7) must satisfy the commutator bracket

$$[D_+, D_-] = -\frac{B}{2}. \qquad (5.8)$$

Defining $k = B\rho^2/2$ and substituting (5.7) into (5.8), the commutator and covariant derivatives fix the eigenvalue of L_1 to be

$$L_1 = \frac{i}{2} k. \qquad (5.9)$$

A Hamiltonian can be written down in terms of the $D_\pm$ as follows,

$$H = -(D_+ D_- + D_- D_+) = -\frac{1}{\rho^2}(L_2 L_3 + L_3 L_2) = -\frac{2}{\rho^2}(L_2 L_3 - i L_1). \qquad (5.10)$$

This can be written in terms of the Casimir operator (4.15) using $L_2 L_3 = -C + iL_1 - L_1^2$ in the following way

$$H = \frac{2}{\rho^2}(C + L_1^2). \qquad (5.11)$$

Using the representation theory in which the eigenvalues of C are $j(j+1)$ and j is taken to be $j = l - k/2$, the associated spectrum of (5.11) is

$$E_l = \frac{2}{\rho^2}\left[\left(l - \frac{k}{2}\right)\left(l - \frac{k}{2} + 1\right) - \frac{k^2}{4}\right]. \qquad (5.12)$$

It is clear that this geometric approach yields useful results both in the nature of fundamental evolution equations and predictions for the trajectories in the classical case, as well as predictions for the energy spectrum and wave functions in the quantum problem. It will also be of interest to extend the work in this study to problems in higher dimensions, and to consider the existence of other classes of solutions to the equations, such as soliton solutions.

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